LETTER TO THE EDITOR

The negative susceptibility of the $n = 0$ vector model and polymer statistics

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Abstract. It is shown, that the negative susceptibility of the $n = 0$ vector model does not imply unphysical characteristics for a related polymer system. In terms of the polymer system it corresponds to the narrowing of the distribution function of the total number of polymers in the solution with respect to the Poisson distribution.

The leading term of the longitudinal susceptibility of an isotropic magnet below the Curie point in a low field is known to be proportional to $h^{-\varepsilon/2}$ where $\varepsilon = 4 - d$ (Vaks *et al* 1968, Patashinskii and Pokrovskii 1973). This is due to long-range transverse correlations of the order parameter, so the leading term is proportional to $n - 1$, i.e. the number of transverse modes of the n-component magnet (Patashinskii and Pokrovoskii 1973). Thus if $n < 1$, the leading term in the longitudinal susceptibility is negative.

It is shown by Des Cloizeaux (1975), that the formal equations for the n-component magnet in the limit of $n = 0$ describe the solution of linear polymers. For arbitrary n there exists a correspondence between the magnet and the solution containing linear and ring polymers (Erukhimovich 1978, Nikamarov and Obukhov 1981). In this case ln n is simply a chemical potential for the number of ring polymers. In this theory the longitudinal correlation function (in the long-wavelength limit it coincides with the longitudinal susceptibility) is related to the correlations between one end of a certain polymer and all the other ends of the polymers in solution. The simple relation given by Schäfer and Witten (1977) is often used.

$$\int \mathrm{d}^d x [\langle \sigma(x)\sigma(0)\rangle - \langle \sigma(0)\rangle^2] = h^2 \chi_{\mathrm{L}} \tag{1}$$

where $\sigma(x)$ is the density of chain end points.

Moore and Wilson (1980) using the identity

$$\int \mathrm{d}^d x [\langle \sigma(x)\sigma(0)\rangle - \langle \sigma(0)\rangle^2] = \frac{1}{V}\left\langle\left(\int \mathrm{d}^d x [\sigma(x) - \langle\sigma(0)\rangle]\right)\right\rangle^2$$

where V is the volume of the system, have concluded that the LHS of (1) is positive, so that the conventional calculations are incorrect for a wide range of polymer concentrations for which $\chi_{\mathrm{L}} < 0$. It must be again noted that it was implied in the derivation of equation (1) that $\langle \sigma(x)\sigma(0)\rangle$ is the correlation function between different end points of polymers. Then $\sigma(x)$ in (1) must be replaced by $\sigma(x) - \delta(x)$. Using the above

identity we obtain

$$\frac{1}{V}\left\langle\left(\int d^d x (\sigma(x) - \langle\sigma(0)\rangle)\right)^2\right\rangle - \langle\sigma(0)\rangle = h^2 \chi_L$$

or

$$\frac{1}{V}[\langle(2N_P - \langle 2N_P\rangle)^2\rangle - \langle 2N_P\rangle] = h^2 \chi_L \tag{2}$$

where N_P is the total number of polymers in the solution. Relation (2) can be derived in a more straightforward way. In the polymer–magnet analogy h plays the role of fugacity for the number of end points of the polymers in the solution. Thus the partition function of the solution can be written in the form (Des Cloiseaux 1975)

$$Z(h) = \sum_{N_P} h^{2N_P} C(N_P) \tag{3}$$

where $C(N_P)$ is the number of different configurations composed of N_P polymers. Calculating the susceptibility $\chi_L = V^{-1} \partial^2 \ln Z / \partial h^2$, we obtain relation (2).

It follows from (2) that the susceptibilities of the formal magnet system related to the solution of polymers can be negative. Instead of $\chi_L \geq 0$, another inequality can be derived. Using $\langle N_P\rangle / V = \frac{1}{2} Mh$, where $M = V^{-1} \partial \ln Z / \partial h$ is the magnetisation of the related magnetic model, we obtain from (2)

$$\chi_L \equiv \frac{\partial M}{\partial h} \geq -\frac{M}{h}.$$

Bearing in mind that M/h is just the transverse susceptibility we obtain

$$\chi_L \geq -\chi_T. \tag{4}$$

This condition holds for any values of h, T and correspondingly for any concentrations of monomers and polymers in the solution.

Let us assume that the distribution of the end points in the solution obeys the statistics of an ideal gas, i.e. is given by the Poisson distribution:

$$P_{N_P} = \cosh\langle 2N_P\rangle \frac{\langle 2N_P\rangle^{2N_P}}{(2N_P)!}. \tag{5}$$

In the normalisation condition $\sum_N P_N = 1$ only even numbers of end points of polymers are included: hence the factor $\cosh\langle 2N_P\rangle$.

The distribution (5) gives exactly

$$\chi_L = 0.$$

If $\chi_L < 0$ it means, that the distribution of the numbers of end points is narrower than the Poisson distribution, since the end points in the regions of the order of the transverse correlation length, i.e. the size of one polymer, are weakly correlated.

To calculate this narrowing of the distribution function we use the first-order perturbation theory result for the susceptibility, given by Moore (1977), written in polymer notations. Only the corrections due to the transverse fluctuations are written here

$$\chi_L = \frac{1}{2vC_m}\left(1 - \frac{2-\varepsilon}{4} R_d vL^{\varepsilon/2}\right) \tag{6}$$

where C_m is the concentration of the monomers, v is the volume of one monomer and L is the mean length of a polymer,

$$R_d = \pi/\sin \tfrac{1}{2}\pi\varepsilon\, \Gamma(\tfrac{1}{2}d)(4\pi)^{d/2}.$$

Using (3) and the relation $h^2 = 2N_P/VL$ we obtain the dispersion

$$\langle N_P^2\rangle - \langle N_P\rangle^2 = \tfrac{1}{2}\langle N_P\rangle + \tfrac{1}{4}\chi_L h^2 V = \tfrac{1}{2}\langle N_P\rangle\left[1 + \frac{1}{2C_m}\left(\frac{1}{vL} - \frac{2-\varepsilon}{4}R_d\frac{L^{\varepsilon/2}}{L}\right)\right]. \tag{7}$$

We see that the narrowing of the distribution function at $d = 3$ is relatively weak, and is of the order of $L^{-1/2}$. However, in the two-dimensional case the situation is more interesting. For $d = 2$ one must take care of the singularity in R_d:

$$\lim_{\varepsilon \to 2} (2 - \varepsilon)R_d = 1/2\pi$$

and

$$\langle N_P^2\rangle - \langle N_P\rangle^2 = \tfrac{1}{2}\langle N_P\rangle\left(1 - \frac{1}{16\pi C_m}\right) \qquad \text{at } L \to \infty. \tag{8}$$

In this case the next order terms of the perturbation theory must be taken into account. This summation of all transverse corrections for a two-dimensional magnet was performed by Polyakov (1975) for arbitrary n. The result for the renormalised magnetic moment in polymer notation is (Nikamarov and Obukhov 1981).

$$\frac{M}{M_0} = \left(1 - \frac{n-2}{4\pi C_m}\ln L\right)^{(n-1)/2(n-2)} \tag{9}$$

Using $\chi_L = \partial M/\partial h$, $h = 2L/M$, we obtain

$$\langle N_P^2\rangle - \langle N_P\rangle^2 = \tfrac{1}{2}\langle N_P\rangle\left[1 + \frac{n-1}{16\pi C_m}\bigg/\left(1 - \frac{n-2}{4\pi C_m}\ln L\right)^{(n-3)/2(n-2)}\right] \tag{10}$$

The longitudinal fluctuations were not taken into account here, as they lead only to the renormalisation of quantities C_m, L in asymptotic equations (7)–(10). If these fluctuations are large the above consideration can be performed in 'blob' units:

$$L^R \sim LC_m^{1/(vd-1)} \qquad C_m^R \sim 1.$$

Thus both at $d = 3$ and $d = 2$ the narrowing of the distribution function is relatively small and vanishes in the asymptotic limit $L \to \infty$.

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